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On the Conjecture of Meinardus on Rational Approximation of e^x, II

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INTRODUCTION

Recently D. J. Newman [4] and the author [1] have established upper and lower bounds for the degree of approximation when e^x is approximated by rational functions on the interval $[-1, +1]$. The bounds differ from the value conjectured by Meinardus [2, p. 168] only by constants. Here we will show that Meinardus' conjecture is true as it stands.

1

This paper is concerned with Meinardus' conjecture on uniform approximation of e^x on the interval $[-1, +1]$ by (m, n) – degree rationals, i.e., by rational functions whose numerator and denominator have degrees m and n, respectively. Meinardus, in his monograph [2, p. 168], conjectured that $E_{m,n}$, i.e., the distance of e^x to the rationals, measured by the sup-norm in $C[-1, +1]$, has the following asymptotic behavior:

$$E_{m,n} = \frac{n!m!}{2^{n+m}(n+m)!\,(n+m+1)!}(1 + o(1)) \qquad \text{as} \quad n + m \to \infty. \quad (1)$$

The crucial point here and in the previous estimates is an observation made by Newman: Let p/q be an (m, n)-degree rational. Given $x \in [-1, +1]$, put $z = (x + iy)/2$ with y real, $x^2 + y^2 = 1$. Then $R(x) = p(z)\,p(\bar{z})/[q(z)\,q(\bar{z})]$ is also an (m, n)-degree rational function in the variable x. Since $e^x = e^z e^{\bar{z}}$, we have

$$e^x - R(x) = 2\,\mathrm{Re}\,e^{\bar{z}}\left(e^z - \frac{p(z)}{q(z)}\right) - \left|\,e^z - \frac{p(z)}{q(z)}\right|^2. \quad (2)$$

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In the sequel only cases are considered where the second term is small when compared with the first one. Moreover assume that $p - e^z q$ has $n + m + 1$ zeros in the domain $|z| \leqslant \frac{1}{2}$ but there q has none; this happens, e.g., when p/q is the Padé approximant. Then by a winding number argument one concludes that $e^x - R(x)$ alternates $(n + m + 1)$-times [1]; one gets a lower bound from the theorem of de la Vallée–Poussin, which together with the obvious upper bound reads

$$2 \min_{|z|=\frac{1}{2}} \left| e^z \left(e^z - \frac{p}{q}\right) \right| \leqslant E_{m,n}(1 + o(1)) \leqslant 2 \max_{|z|=\frac{1}{2}} \left| e^z \left(e^z - \frac{p}{q}\right) \right| . \qquad (3)$$

Therefore we are interested in constructing rationals p/q such that $|e^z(e^z - p/q)|$ varies only little on the circle $|z| = \frac{1}{2}$.

2

We will start with the (m, n)-Padé approximant which is given by

$$p(z) = \int_0^\infty t^n (t + z)^m e^{-t} \, dt, \qquad q(z) = \int_0^\infty (t - z)^n t^m e^{-t} \, dt. \qquad (4)$$

First we derive an asymptotic property of the remainder term

$$\begin{aligned} p(z) - e^z q(z) &= \int_0^\infty t^n (t + z)^m e^{-t} \, dt - \int_0^\infty (t - z)^n t^m e^{-t+z} \, dt \\ &= \int_0^{-z} t^n (t + z)^m e^{-t} \, dt \\ &= (-1)^{n+1} z^{n+m+1} \int_0^1 u^n (1 - u)^m e^{uz} \, du. \end{aligned}$$

After inserting the Taylor's series for the exponential function the integral above becomes

$$\begin{aligned} &\int_0^1 u^n (1 - u)^m \sum_{k=0}^\infty \frac{1}{k!} u^k z^k \, du \\ &= \sum_{k=0}^\infty \frac{m!(n + k)!}{(n + m + k + 1)!} \frac{z^k}{k!} \qquad (5) \\ &= \frac{m!n!}{(n + m + 1)!} \sum_{k=0}^\infty \frac{(n + 1)_k (n + m)^k}{n^k (n + m + 2)_k} \frac{1}{k!} \left(\frac{n}{n + m} z\right)^k . \end{aligned}$$

Here the convention $(a)_k := a(a+1)\cdots(a+k-1)$ known from hypergeometric functions is used. Now we recall a simple observation on sequences of power series. Let

$$\psi_\nu(z) = \sum_{k=0}^{\infty} a_{k\nu} \frac{z^k}{k!}, \qquad \nu = 1, 2, \ldots,$$

with $|a_{k\nu}| \leqslant 1$ and $\lim a_{k\nu} = 0$ for each k. Then $\sup\{|\psi_\nu(z)|, |z| \leqslant 1\}$tends to zero. From (5) we conclude

$$e^z q(z) - p(z) = \frac{m!n!}{(m+n+1)!}(-1)^{n+1} z^{n+m+1} e^{(n/(n+m))z}(1 + \psi_1(z)), \quad (6)$$

where $\psi_1(z)$, $[z] < 1$, becomes arbitrarily small as $n + m \to \infty$. To be more specific, from (5) first a small additive correction to the exponential function is obtained which is changed into a multiplicative term.

3

In the same spirit from (4) an asymptotic formula for q is derived after the binomial formula is applied to $(t-z)^n$:

$$q(z) = (n+m)!\, e^{-(n/(n+m))z}(1 + \psi_2(z)). \quad (7)$$

For an estimate of the correction term, $|\psi_2(z)| \leqslant |z|^2 e^{2|z|}/(2m+2n)$, the reader is referred to [5, p. 248]. From (6) and (7) it follows that

$$e^z \left(e^z - \frac{p(z)}{q(z)}\right) = \frac{n!m!(-1)^n}{(n+m)!\,(n+m+1)!} z^{n+m+1} e^{\alpha z}(1 + o(1)) \quad (8)$$

where $\alpha = 1 + 2n/(n+m)$. We note that $1 \leqslant \alpha \leqslant 3$.

4

The function (8) [more precisely its modulus] is not constant on the circle $|z| = \frac{1}{2}$ mainly because $e^{\alpha z} z^{n+m+1}$ is not. On the other hand by choosing z_0 appropriately we can achieve that

$$|e^{\alpha z}(z - z_0)^{n+m+1}| \quad (9)$$

deviates very little from a constant on that circle.

It is easily checked by some elementary calculations that

$$e^{-2/(3t)} \leqslant \left| e^z \left(z - \frac{1}{t}\right)^t \right| \leqslant e^{+2/(3t)}, \qquad |z| = 1,\ t \geqslant 3.$$

After taking the $(\alpha/2)$th power of these inequalities, with z replaced by $2z$ and with $N = \alpha t/2$ we get

$$e^{-\alpha^2/(6N)} \leqslant 2^N \left| e^{\alpha z} \left(z - \frac{\alpha}{4N}\right)^N \right| \leqslant e^{+\alpha^2/(6N)}, \qquad |z| = \frac{1}{2}.$$

Consequently, when putting $N = n + m + 1$, $z_0 = \alpha/[4(n + m + 1)]$ we get a close-to-circularity property for (9).

5

Let z_0 be as above. Then $\tilde{p}(z)/\tilde{q}(z) = e^{z_0} p(z - z_0)/q(z - z_0)$ is the Padé approximant to e^z at z_0. From

$$e^z \left(e^z - \frac{\tilde{p}(z)}{\tilde{q}(z)}\right) = \frac{n!m!(-1)^n}{(n+m)!\,(n+m+1)!} (z - z_0)^{n+m+1} e^{\alpha z}(1 + o(1)) \qquad (10)$$

where $(1 + o(1)) = e^{(2-\alpha)z_0}(1 + \psi_1(z - z_0))(1 + \psi_2(z - z_0))^{-1}$, it follows that Meinardus' conjecture is true.

The result remains true if the approximation problem with the weight function $w(x) = e^{-x}$ is considered [3]. Only the exponent α has to be replaced by $\alpha - 2$. In each case a rational function is constructed such that the error curve is near to a circular one [6].

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Note added in proof. For analogous investigations of rational approximation of $\sqrt{x}$ the reader is referred to [7].

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